AMS-311. Spring 2005. Homework 6. Topics: Continuous random variables. Derived distributions. Conditional expectation and variance. Transforms.

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1). Let X be a standard normal random variable. Find the PDF of the random variables $Y = 3X - 1$ and $Z = 3X^2 - 1$.

2). Let X be a random variable with PDF $f_X(x)$. Find the PDF of the random variable $Y = |X|$

(a) when $f_X(x) = 1/3$, $-2 \leq x \leq 1$

(b) when $f_X(x) = 2e^{-2x}$, $x > 0$

(c) for general $f_X(x)$

3). Let X have a uniform distribution in the unit interval $[0, 1]$, and let Y have an exponential distribution with parameter $\lambda = 2$. Assume that X and Y are independent. Let $Z = X+Y$.

(a) Find $P(Y \geq X)$.

(b) Find the conditional PDF of Z given that $Y = y$.

(c) Find the conditional PDF of Y given that $Z = 3$.

4). Let X and Y be the random variables with the following joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} 2, & x \geq 0, y \geq 0, \text{ and } x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Calculate $\mathbf{E}[XY]$.

(b) **[Extra credit]** Calculate the PDF of Y/X.

5). (a) Find the transform of the discrete uniform random variable X, taking values from 1 to n with equal probabilities.

(b) Find the transform of the continuous uniform random variable Y, taking values from the interval $[a, b]$.

6). Let X be a random variable such that

$$M_X(s) = a + be^{2s} + ce^{4s}, \quad \mathbf{E}[X] = 3, \quad \mathbf{var}(X) = 2.$$

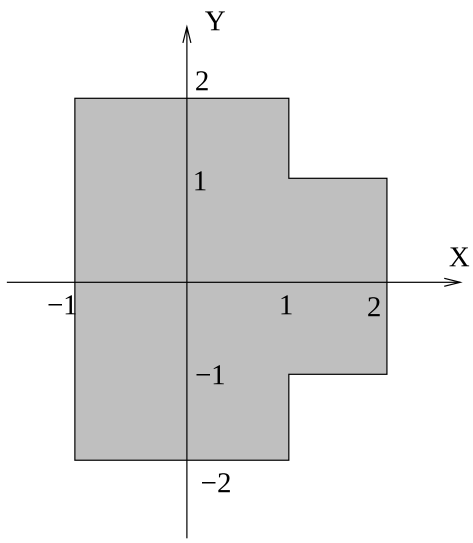
Find a, b, and c, and the PMF of X.

7). The transform and the mean of a random variable X are given by $M_X(s) = ae^s + be^{13(e^s-1)}$ and $\mathbf{E}[X] = 5$ respectively. Determine the numerical values of:

(a) The constants a and b.
(b) $\mathbf{E}\left[e^{5X}\right]$.
(c) $\mathbf{E}[X^2]$.
(d) **[Extra credit]** $P(X = 1)$.

8). Independent random variables X and Y have PDFs whose transforms are $M_X(s)$ and $M_Y(s)$. Random variable R is defined to be $R = X + Y$. Use $M_R(s)$ and the moment generating properties of transforms to show that $\mathbf{E}[R] = \mathbf{E}[X] + \mathbf{E}[Y]$ and $\mathbf{var}(R) = \mathbf{var}(X) + \mathbf{var}(Y)$.

9). Random variables X and Y have the joint PDF $f_{X,Y}(x,y) = 0.1$ in the region, shown below:



(a) Find the conditional PDFs $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$, for various values of x and y, respectively.
(b) Find $\mathbf{E}[X|Y], \mathbf{E}[X]$, and $\mathbf{var}(X|Y)$. Use these to calculate $\mathbf{var}(X)$.
(c) Find $\mathbf{E}[Y|X], \mathbf{E}[Y]$, and $\mathbf{var}(Y|X)$. Use these to calculate $\mathbf{var}(Y)$.

10). Widgets are packed into cartons which are packed into crates. The weight (in pounds) of a widget, X, is a continuous random variable with PDF

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The number of widgets in any carton, K, is a random variable with PMF

$$p_K(k) = \frac{\mu^k e^{-\mu}}{k!}, \quad k = 0, 1, 2, \ldots.$$

The number of cartons in a crate, N, is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p), \quad n = 1, 2, 3, \ldots.$$

Random variables X, K, and N are mutually independent. Determine

(a) **[Extra credit]** The probability that a randomly selected crate contains exactly one widget.

(b) The expected value and variance of the number of widgets in a crate.

(c) The transform or the PDF for the total weight of the widgets in a crate.

(d) The expected value and variance of the total weight of the widgets in a crate.

11). **[Extra credit]** Let $X_1, \ldots, X_n$ be independent random variables that are uniformly distributed in the interval $[0, 1]$. Let S and L be the smallest and largest, respectively, of $X_1, \ldots, X_n$. Determine the joint PDF of S and L.

12). **[Extra credit]** Let random variables X and Y have the bivariate normal PDF

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right\}, \quad -\infty < x, y < \infty$$

where ρ denotes the correlation coefficient between X and Y.

(a) Determine the numerical values of $\mathbf{E}[X], \mathbf{var}(X), \mathbf{E}[Y]$, and $\mathbf{var}(Y)$.

(b) Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent normal random variables, and determine $\mathbf{E}[Z]$ and $\mathbf{var}(Z)$.

(c) Deduce that

$$P\left(\{X > 0\} \bigcap \{Y > 0\}\right) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho.$$

13). **[Extra credit]** Consider the following game. We have a fair six-sided die and two players in a room. The players take turns rolling the die. If a die comes up with n, the player is given n turns at the Gauss machine. The Gauss machine generates rewards (note that the rewards can be negative!) which are independent identically distributed normal random variables with mean m and variance σ^2. At any time, the score of a player is the total of the outputs he has received from the Gauss machine. It takes each player 1 minute to roll the die and each turn at the Gauss machine takes 1 minute. There are no other delays in the game. We consider that a play starts with the rolling of the die and ends with the next roll of the die.

(a) Find the PDF or transform for the score of player 1 right before he rolls the die for the $(k+1)$-th time.

(b) We change the game so that each play consists of rolling the die once and getting one turn at the Gauss machine. The score for the play is obtained by multiplying the output of the Gauss machine by the number of points on the die roll. Find the PDF or transform for the score of player 1 right before he rolls the die for the $(k+1)$-th time.